Freak Waves and Analytical Theory of Wind-Driven Sea

V.E. Zakharov

There are two types of rare catastrophic events on the ocean surface:

1. Freak waves (major catastrophic event)
2. Wave breaking (minor catastrophic event)

Freak waves are responsible for ship-wreaking, loss of boats, cargo and lives. Wave breaking is the most important mechanism of wave energy dissipation and for transport of momentum from wind to ocean.

Analytic theory for both of them is not developed

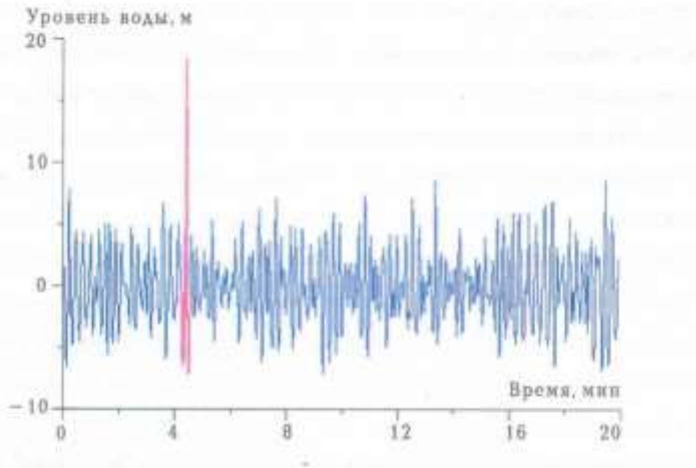


Рис. 1.9. «Новогодняя волна», зарегистрированная в Северном море 1 января 1995 г. [106]

“New Year” wave – 1995 year

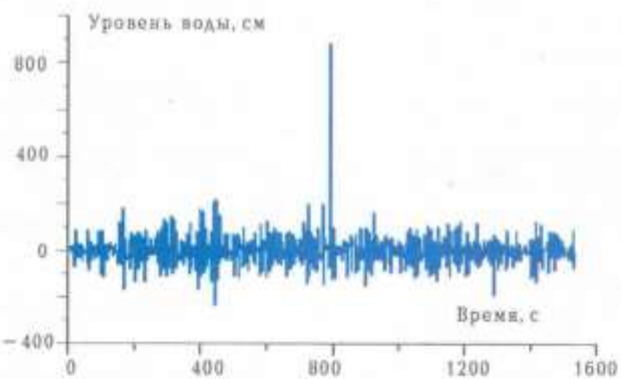
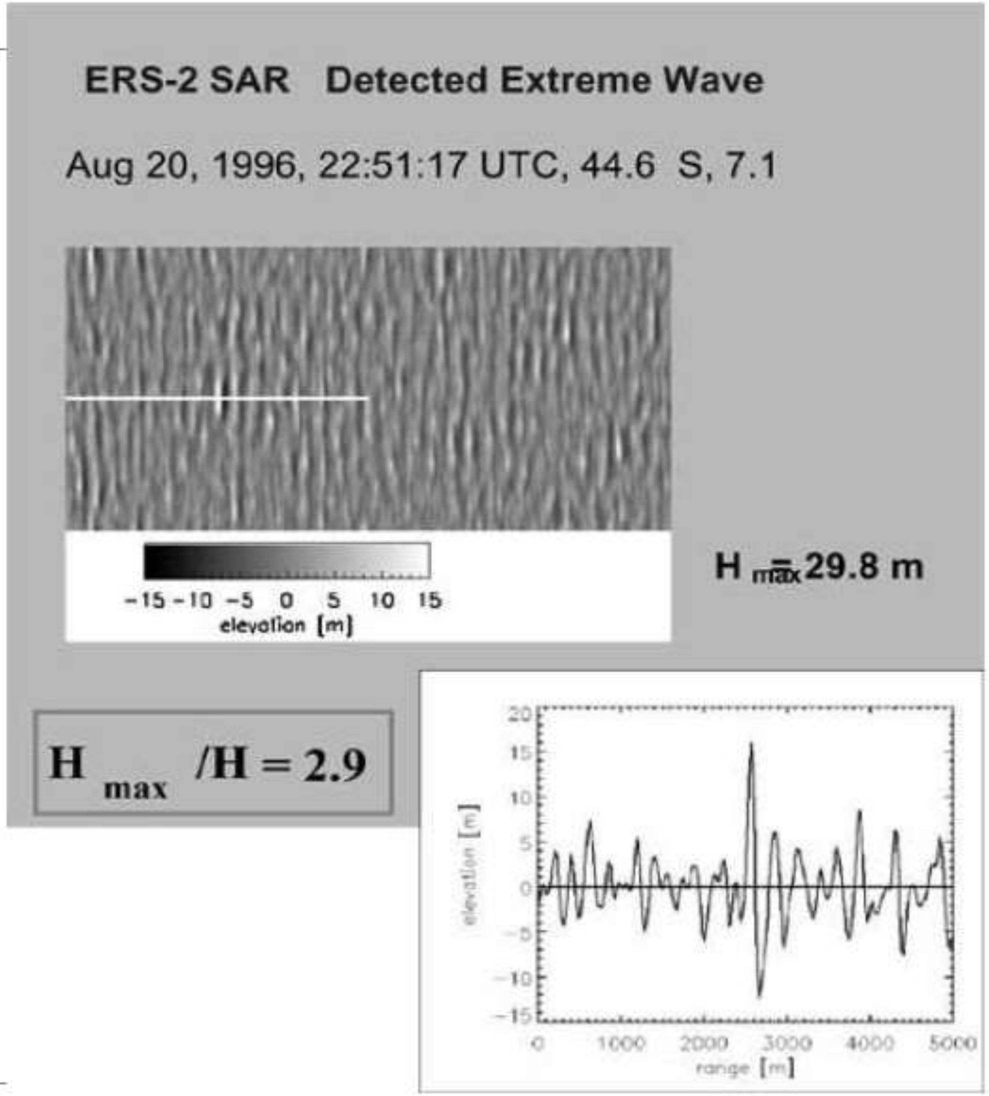


Рис. 1.10. Аномальная волна, зарегистрированная с буя в Черном море 22 ноября 2001 г. [12, 30]

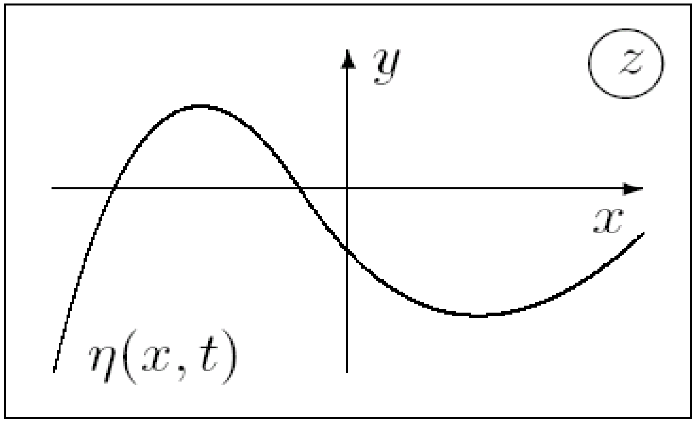
Extreme wave in the Black sea – 2002 year

Satellite Radar




WILSTAR

Equations



potential irrotational flow

$$\triangle\phi(x,y,t)=0$$

Boundary conditions: $\left[\begin{array}{l}\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+g\eta=\frac{P}{\rho},\\ \frac{\partial\eta}{\partial t}+\eta_x\phi_x=\phi_y\end{array}\right|$ at $y=\eta(x,t)$.

Conformal Mapping

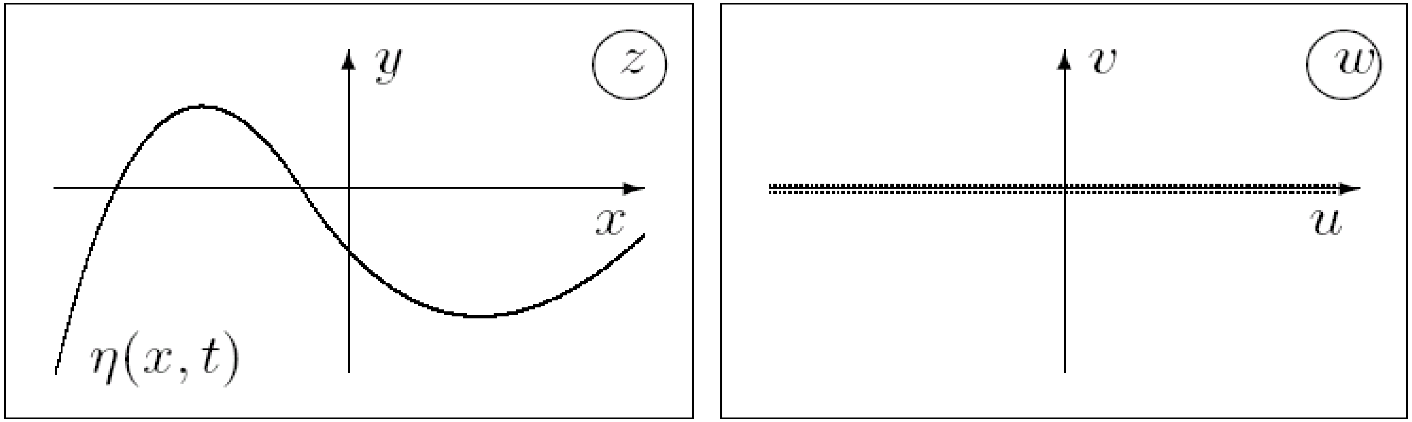
of the domain on the plane $z = x + iy$,

$$-\infty < x < \infty, \quad -\infty < y \leq \eta(x,t)\,,$$

to the lower half-plane,

$$-\infty < u < \infty, \quad -\infty < v \leq 0\,,$$

on the plane $\omega = u + iv$.



Conformal Mapping

After this mapping, the surface profile is given parametrically by

$$y = y(u,t), \quad x = u + \tilde{x}(u,t)\,.$$

Functions y and $\tilde{x}$ are coupled by the relations:

$$y = \hat{H}\tilde{x} \qquad\qquad \tilde{x} = -\hat{H}y\,.$$

Here $\hat{H}$ is the Hilbert transformation,

$$\hat{H}f(u) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(u')}{(u'-u)} du'\,.$$

For Fourier harmonics $y_k = i$sign$(k) x_k$.

Cubic Equations

It turned out, that the equations can be simplified just by changing variables. Introduce instead of $Z(w,t)$ and $\Phi(w,t)$ another analytic functions $R(w,t)$ and $V(w,t)$

$$R = \frac{1}{Z_w}, \qquad \Phi_w = -iVZ_w.$$

$$\begin{aligned} R_t &= i\left[U R' - U' R\right], \\ V_t &= i\left[UV' - R\hat{P}(V\bar{V})'\right] + g(R-1). \end{aligned}$$

Complex transport velocity U is defined via $\hat{P}$

$$U = \hat{P}(V\bar{R} + \bar{V}R).$$

$$\eta = \eta_0(x,t)(1+\epsilon e^{i(\Omega t + px)})$$

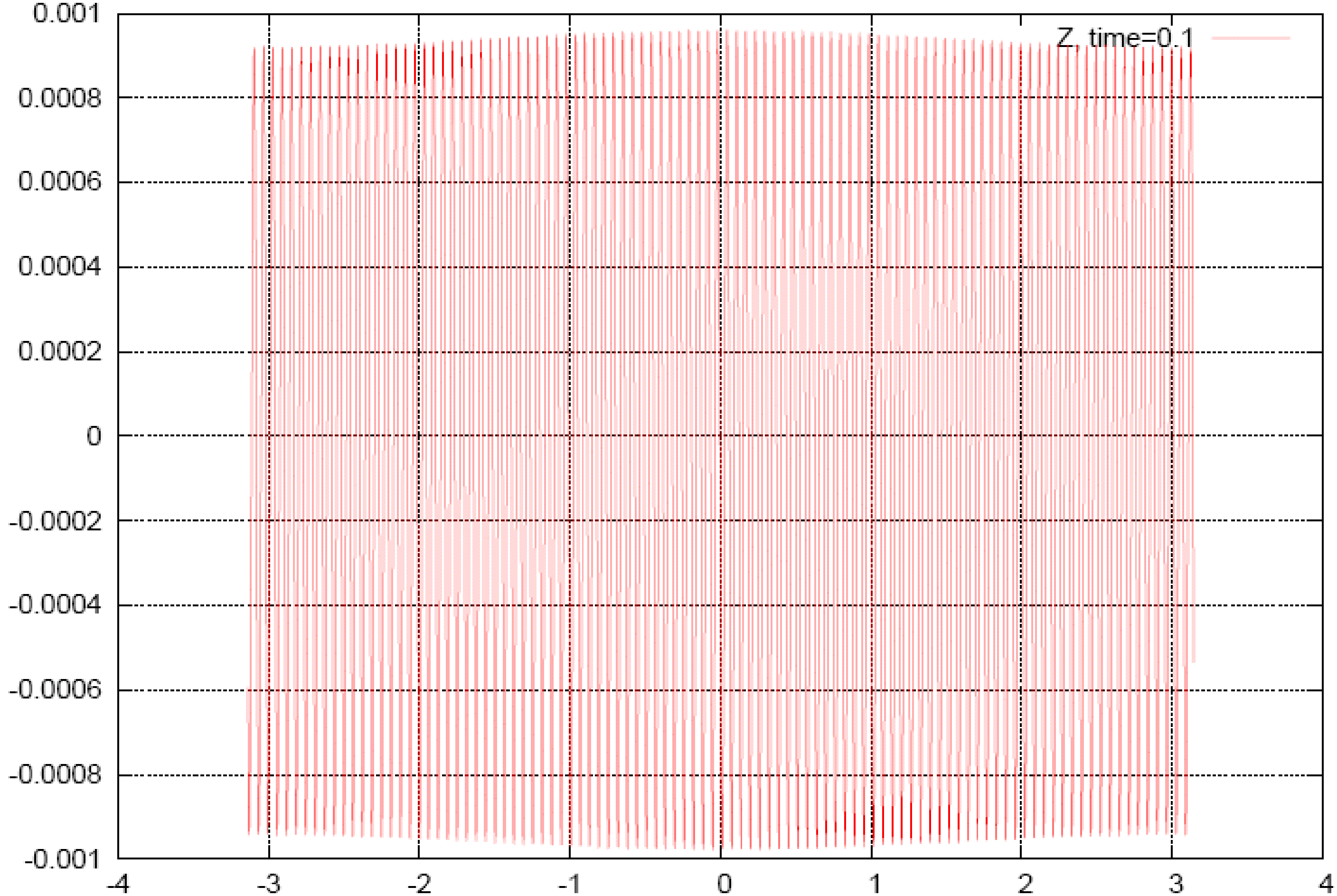
$$\frac{\Omega^2}{\omega_k^2} = \frac{1}{4}(-(pa)^2 + \frac{1}{4}(\frac{p}{k})^4)$$

Modulation instability

- Lighthill - 1965
- Zakharov - 1966
- Benjamin, Fair - 1967 B-F made experiment!

100 waves $\mu = 0.095$

* Put 100 such waves with small perturbation in the periodic domain of 2π



100 waves $\mu = 0.095$

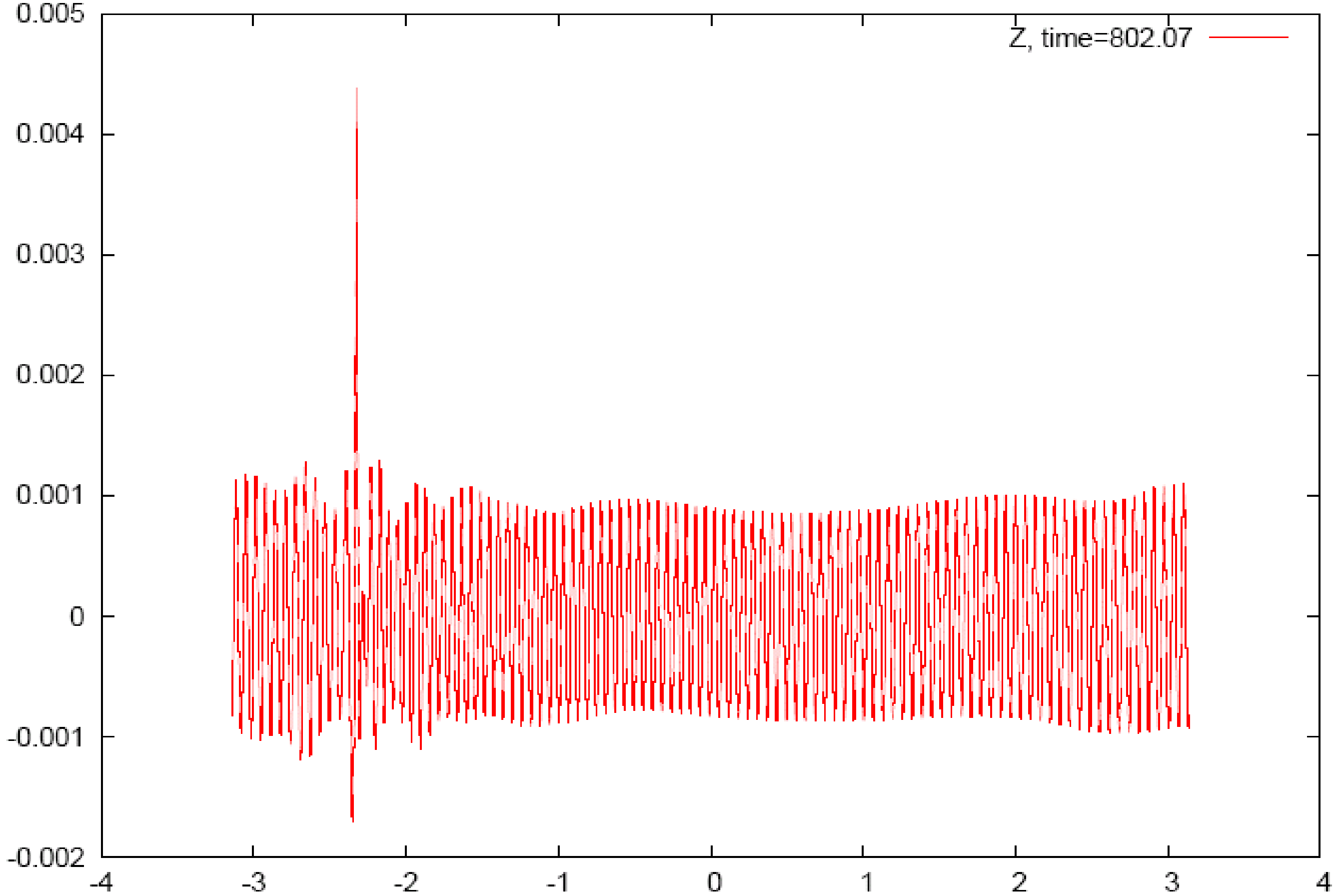


Figure 6: Surface profile

100 waves $\mu = 0.095$

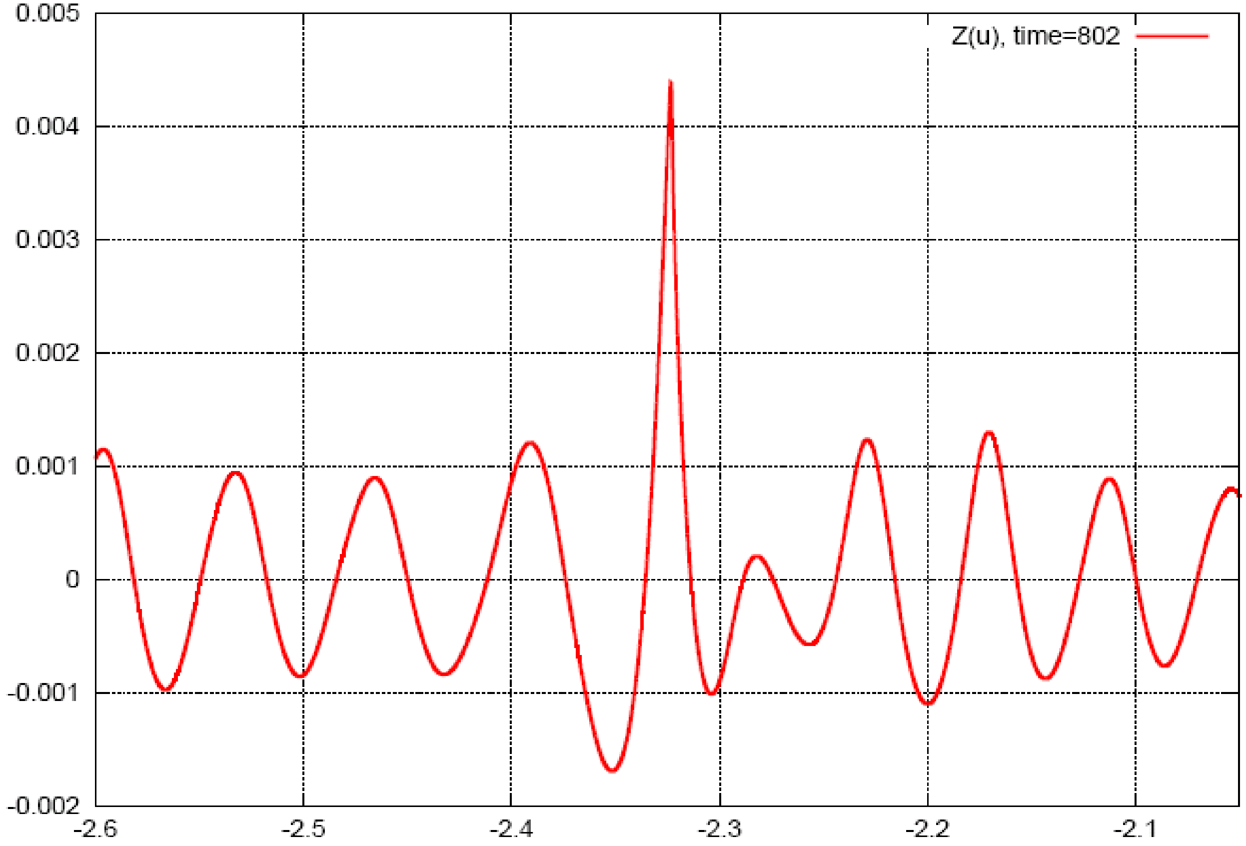


Figure 7: Zoom in surface profile

Wave breaks

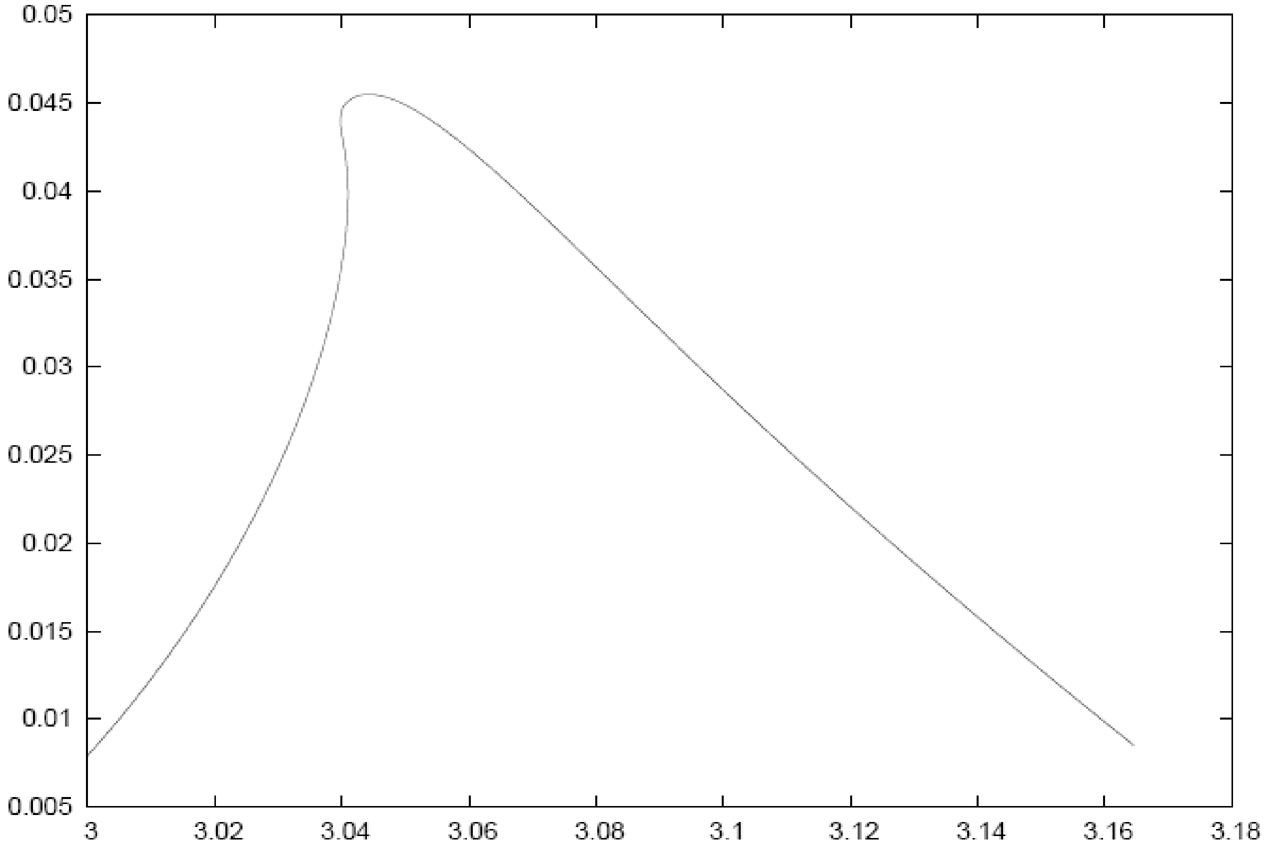
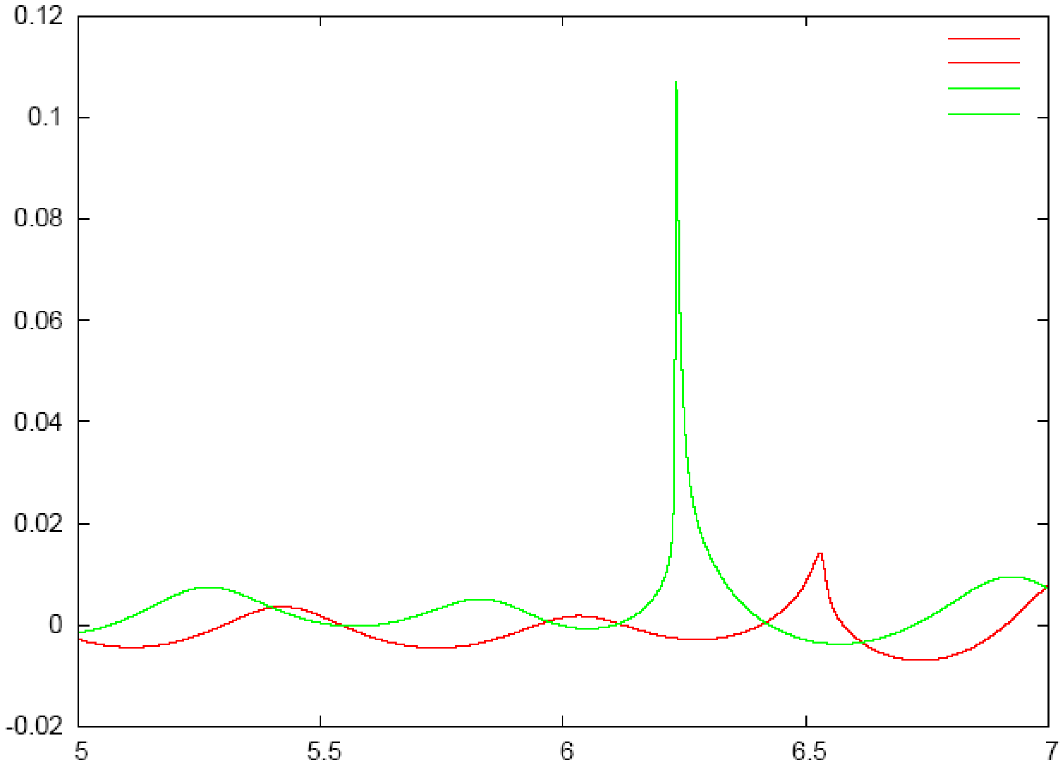


Figure 9: Overturning

Wave breaks. Momentum



NLSE Soliton $\mu = 0.1$

$$R(u) = 1 + \mu \frac{e^{-ik_0 u}}{\cosh(L_0 k_0 u)}, \qquad V(u) = -i\sqrt{\frac{g}{k_0}}\mu \frac{e^{-ik_0 u}}{\cosh(L_0 k_0 u)},$$

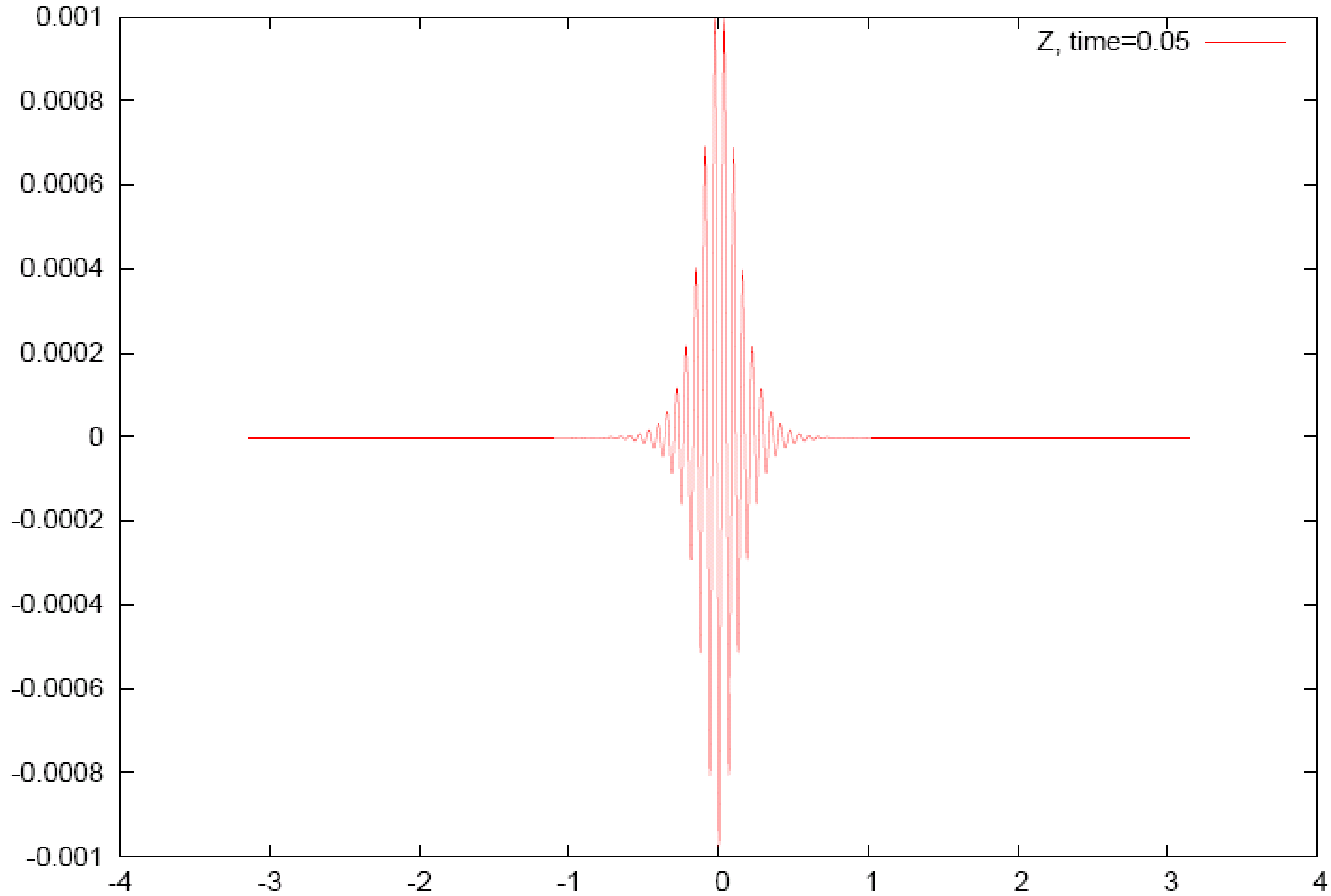
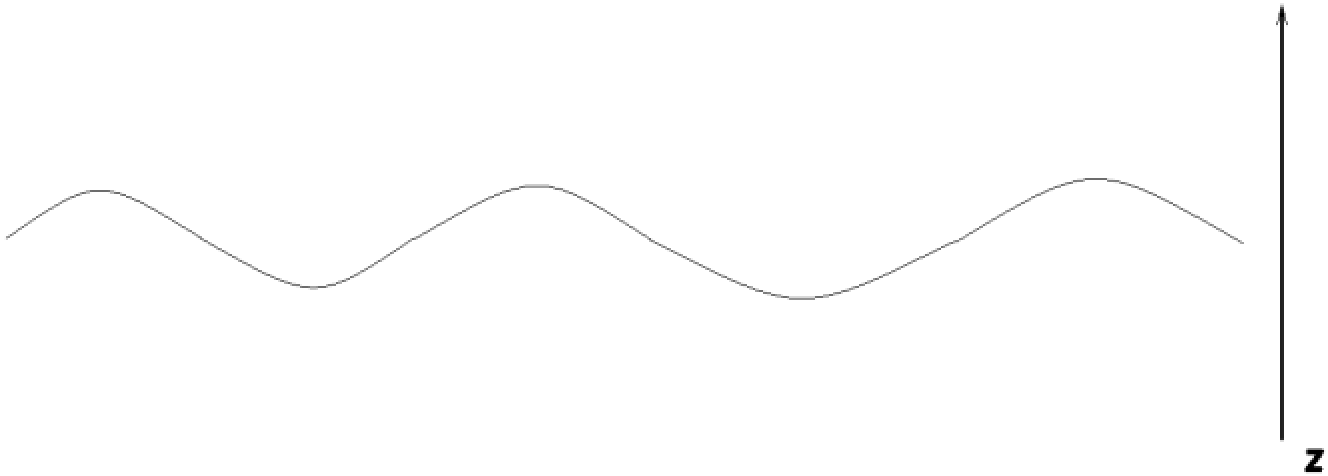


Figure 10: Initial soliton. $k_0 = 100$, $L_0 = 0.1$.

Old is always gold

Sir Francis Beaufort, FRS, FRO, FRGS,
1774, Ireland — 1857, Sussex

The Beaufort Scale is an empirical measure for describing wind speed based mainly on observed sea conditions. Its full name is the Beaufort Wind Force Scale.



$$-\infty < Z < \eta(r, z) \qquad r = (x, y)$$

$$V = \nabla\Phi \qquad divV = 0 \qquad \Delta\Phi = 0$$

$$\Psi = \Phi\,|_{z=h} \qquad \eta_t = \frac{\delta H}{\delta \Psi} \qquad \Psi_t = -\frac{\delta H}{\delta \eta}$$

$$H = T + U$$

$$T = \frac{1}{2}\int_r dr \int_{-\infty}^{\eta} \left(\nabla\Phi\right)^2 dz = -\int G(s,s')\Psi(s)\Psi(s')ds ds'$$

$G(s,s') = G(s',s)$ - Green function of the Dirichlet-Neuman problem

$$\Phi = \Psi\,|_{z=h} \qquad \frac{\partial\Phi}{\partial z} \to 0 \qquad z \to \infty$$

$$H = \underset{\mu^2}{H_0} + \underset{\mu^3}{H_1} + \underset{\mu^4}{H_2} + \ldots$$

$\mu \cong k\eta$ -- average steepness

Truncated equations:

$$\eta_t = \hat{k}\psi - (\nabla(\eta\nabla\psi)) - \hat{k}[\eta\hat{k}\psi] + \hat{k}(\eta\hat{k}[\eta\hat{k}\psi]) + \tfrac{1}{2}\Delta[\eta^2\hat{k}\psi] + \tfrac{1}{2}\hat{k}[\eta^2\Delta\psi]$$

$$\psi_t = -g\eta - \tfrac{1}{2}[(\nabla\psi)^2 - (\hat{k}\psi)^2] - [\hat{k}\psi]\hat{k}[\eta\hat{k}\psi] - [\eta\hat{k}\psi]\Delta\psi$$

Normal variables:

$$\eta_k = \sqrt{\frac{\omega_k}{2g}}\left(a_k + a_k^*\right)$$

$$\Psi_k = i\sqrt{\frac{\omega_k}{2|k|}}\left(a_k - a_k^*\right)$$

$$\frac{\partial a_k}{\partial t} = i\frac{\delta H}{\delta a^*}$$

Canonical transformation - eliminating three-wave interactions:

$$a_0 = b_0 +$$

$$+ \int A^{(1)}_{012} b_1 b_2 \delta_{0-1-2} dk_{12} + \int A^{(2)}_{012} b^*_1 b_2 \delta_{0+1-2} dk_{12} + \int A^{(3)}_{012} b^*_1 b^*_2 \delta_{0+1+2} dk_{12}$$

$$+ \int B^{(1)}_{0123} b_1 b_2 b_3 \delta_{0-1-2-3} dk_{123} + \int B^{(2)}_{0123} b^*_1 b_2 b_3 \delta_{0+1-2-3} dk_{123} +$$

$$+ \int B^{(3)}_{0123} b^*_1 b^*_2 b_3 \delta_{0+1+2-3} dk_{123} + \int B^{(4)}_{0123} b^*_1 b^*_2 b^*_3 \delta_{0+1+2+3} dk_{123} + O(b^4)$$

$$H \Rightarrow \int \omega_k b_k b^*_k dk + \frac{1}{2} \int T_{k k_1 k_2 k_3} b^*_k b^*_{k_1} b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3}$$

$$T(\varepsilon k, \varepsilon k_1, \varepsilon k_2, \varepsilon k_3) = \varepsilon^3 T(k, k_1, k_2, k_3)$$

$$
\begin{aligned}
T_{1234} = -\frac{1}{32\pi^2}\frac{1}{(q_1q_2q_3q_4)^{1/4}} \Big\{ & -12 q_1 q_2 q_3 q_4 \\
& - 2(\omega_1+\omega_2)^2 \left[\omega_3\omega_4(k_1k_2 - q_1q_2) + \omega_1\omega_2(k_3k_4 - q_3q_4)\right] \\
& - 2(\omega_1-\omega_3)^2 \left[\omega_2\omega_4(k_1k_3 + q_1q_3) + \omega_1\omega_3(k_2k_4 + q_2q_4)\right] \\
& - 2(\omega_1-\omega_4)^2 \left[\omega_2\omega_3(k_1k_4 + q_1q_4) + \omega_1\omega_4(k_2k_3 + q_2q_3)\right] \\
& + (k_1k_2 + q_1q_2)(k_3k_4 + q_3q_4) \\
& + (-k_1k_3 + q_1q_3)(-k_2k_4 + q_2q_4) \\
& + (-k_1k_4 + q_1q_4)(-k_2k_3 + q_2q_3) \\
& + 4(\omega_1+\omega_2)^2 \frac{(k_1k_2 - q_1q_2)(k_3k_4 - q_3q_4)}{q_{1+2} - (\omega_1+\omega_2)^2} \\
& + 4(\omega_1-\omega_3)^2 \frac{(k_1k_3 + q_1q_3)(k_2k_4 + q_2q_4)}{q_{1-3} - (\omega_1-\omega_3)^2} \\
& + 4(\omega_1-\omega_4)^2 \frac{(k_1k_4 + q_1q_4)(k_2k_3 + q_2q_3)}{q_{2-3} - (\omega_1-\omega_4)^2} \Big\}
\end{aligned}
$$

where $q = |k|$

Statistical theory of wind-driven seas

- The Hasselmann equation (1962) - kinetic equation for water waves

$$\left. dE \middle/ dt \right. = \boxed{S_{in} + S_{diss}} + \boxed{S_{nl}}$$

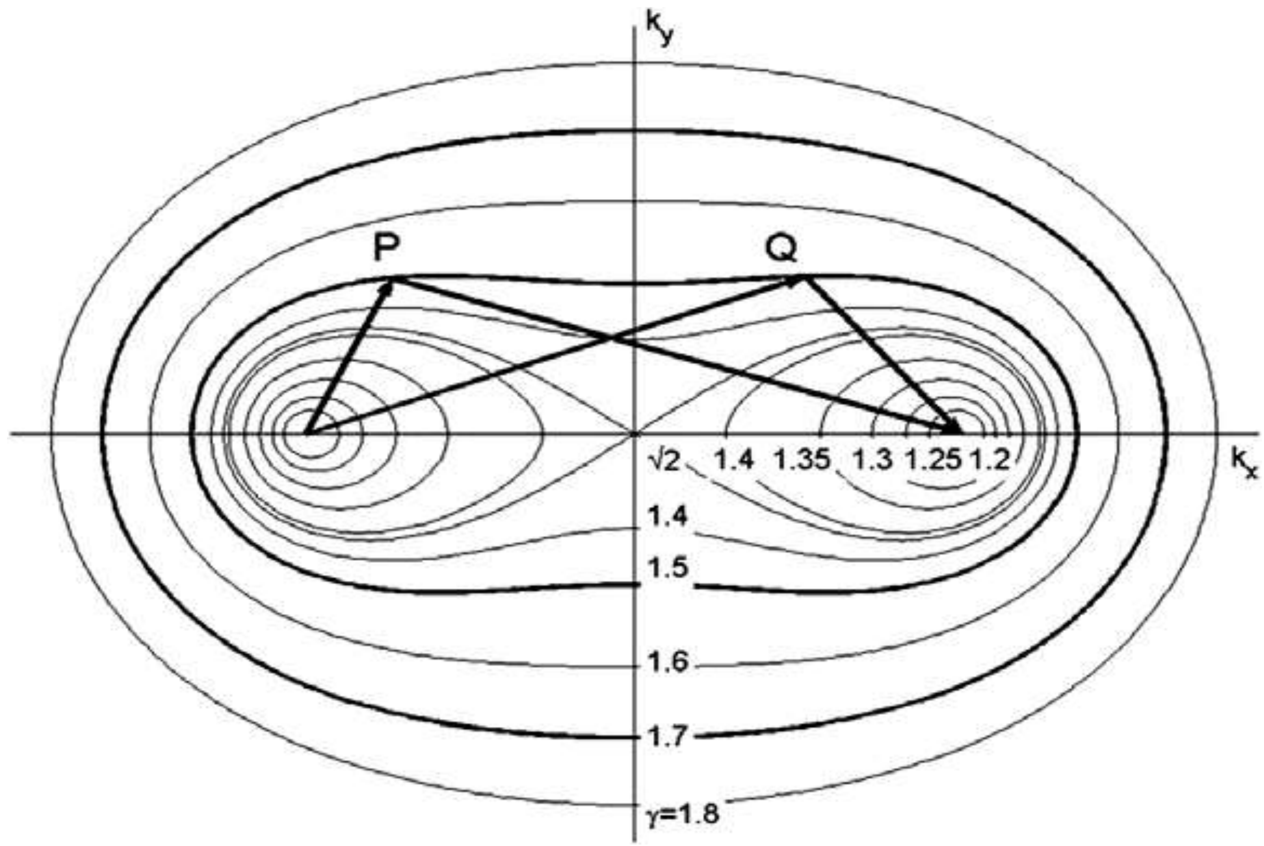
$S_{in}, \quad S_{diss}$ - empirical functions

S_{nl} - the `first principle' term

$$\boxed{\begin{aligned} S_{nl} = 2\pi \int |T_{0123}|^2 (n_0 n_2 n_3 + n_1 n_2 n_3 - n_0 n_1 n_2 - n_0 n_1 n_3) \\ \times \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3) \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \end{aligned}}$$

Interaction coefficients and resonance conditions

4-wave resonance curves



Is there a chance for an analytical theory?

Homogeneity properties

$$S_{nl}\left[\alpha N(\beta\mathbf{k})\right]=\alpha^{3}\beta^{19/4}S_{nl}\left[N(\mathbf{k})\right]$$

Exact stationary solutions

$$E=C_{p}g^{4/3}P^{1/3}\omega^{-4}$$

- direct cascade
Zakharov & Filonenko 1966

$$E=C_{p}g^{4/3}Q^{1/3}\omega^{-11/3}$$

- inverse cascade
Zakharov & Zaslavskii 1982

Phillips, O.M., JFM. V.156,505-531, 1985.

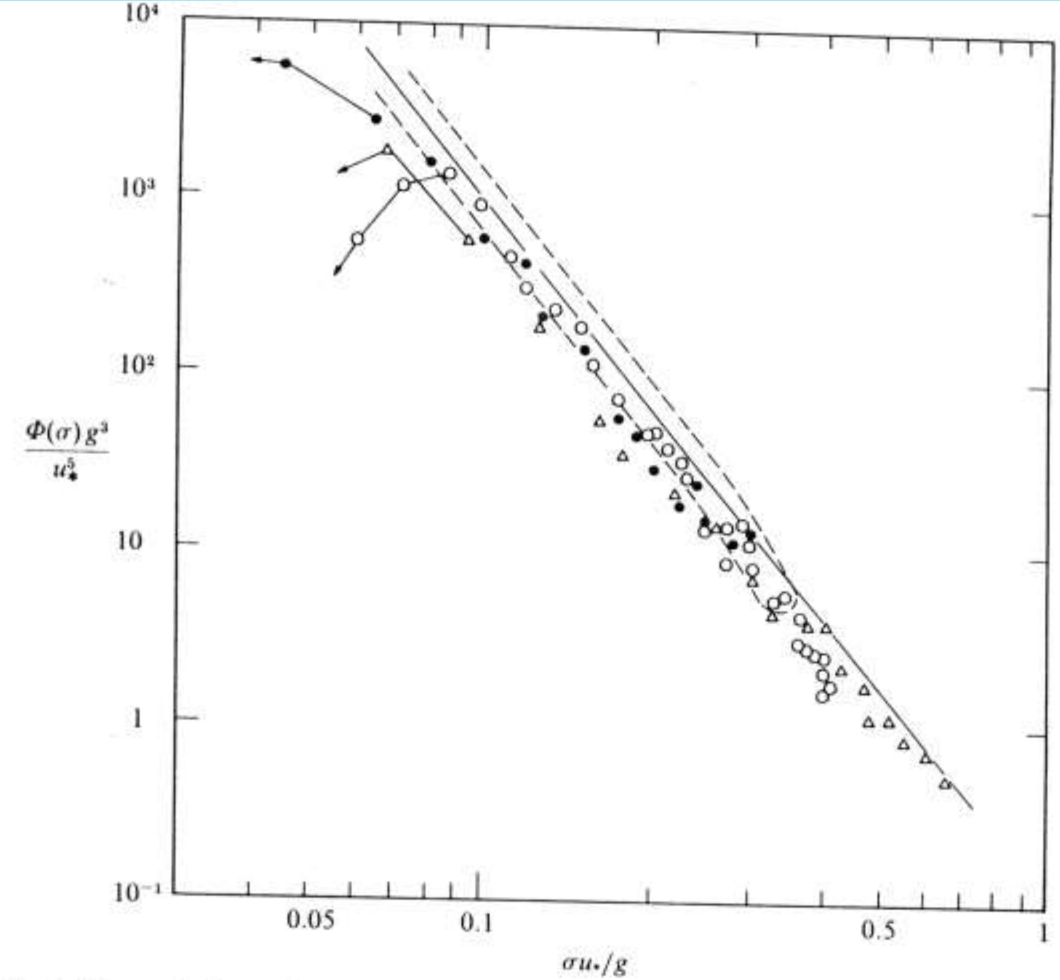
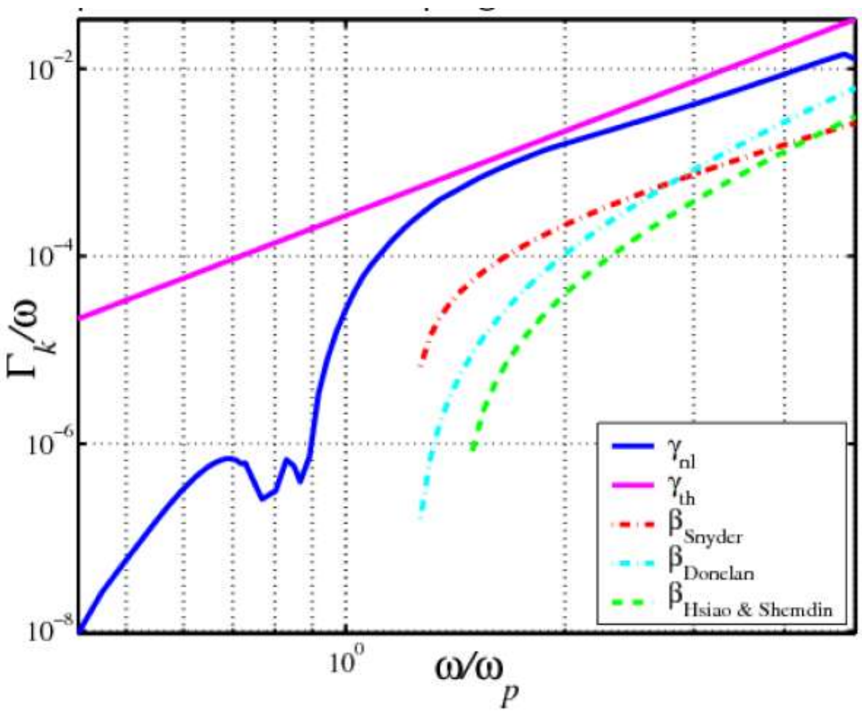


FIGURE 3. A dimensionless plot of the frequency spectrum in the equilibrium range. The broken lines enclose the extensive measurements of Forristall (1981) and the continuous line represents (3.4) with $\alpha = 0.11$. The open circles indicate results from Kawai *et al*. (1977) at $u_* = 37.2$ cm/s; results from Kondo *et al*. (1973) are shown as solid circles at $u_* = 49.6$ cm/s and the triangles at $u_* = 24.4$ cm/s.

The nonlinearity gives a chance for the analytical theory



The nonlinear relaxation rate is one (or more) orders higher than wind wave pumping rate

Thus, an asymptotic model can be developed where effect of wave-wave resonant interactions is a dominating mechanism

Self-similar solutions

$$E(\mathbf{k}) = ax^{p+4q}\Phi_p(b\mathbf{k}x^{2q})$$

Power-like dependencies for total energy E and a characteristic frequency σ

$$E = \varepsilon_0 x^p$$

$$\omega_p = \omega_0 x^{-q}$$

To check in simulations?

`Magic links' for the SS-solutions

Linear links of exponents

$$p_\tau = \frac{9q_\tau - 1}{2}; \quad p_\chi = \frac{10q_\chi - 1}{2}$$

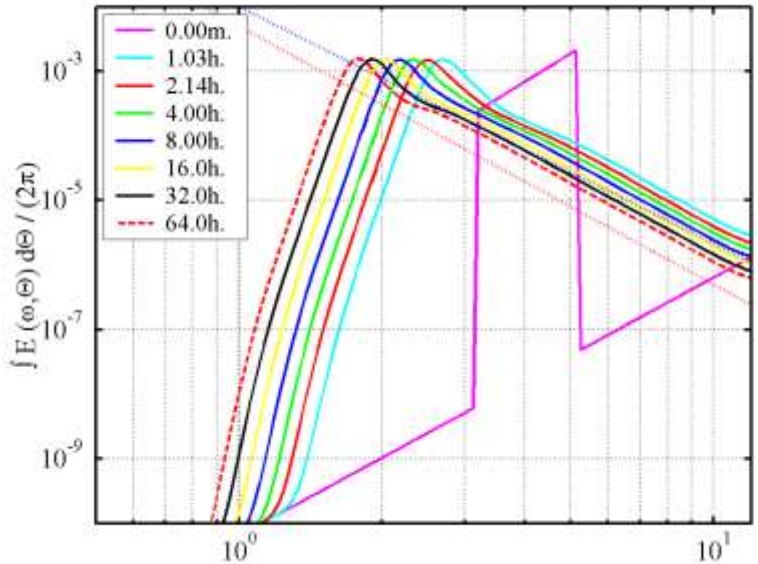
Kolmogorov-like link of energy and its flux - pre-exponents

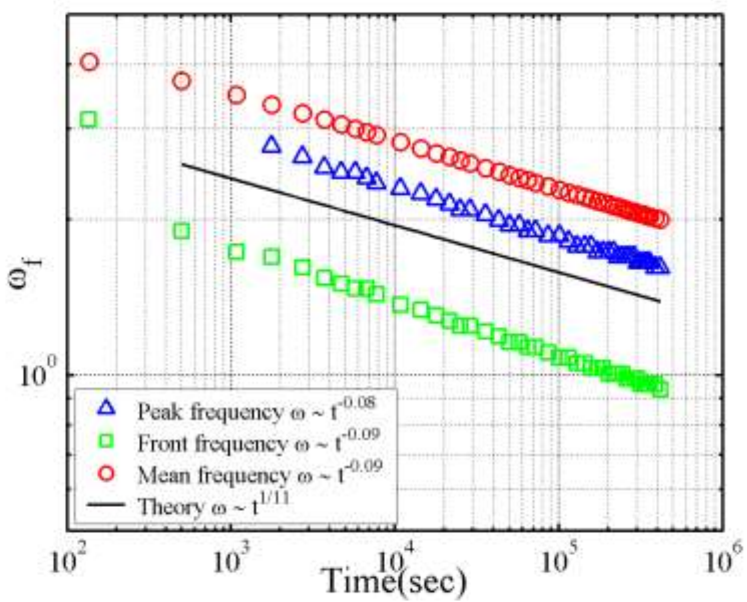
$$\frac{E\omega_p^4}{g^2} = \alpha_{ss}\left(\frac{\omega_p^3 \, {}^{dE}\!/_{dt}}{g^2}\right)^{1/3}$$

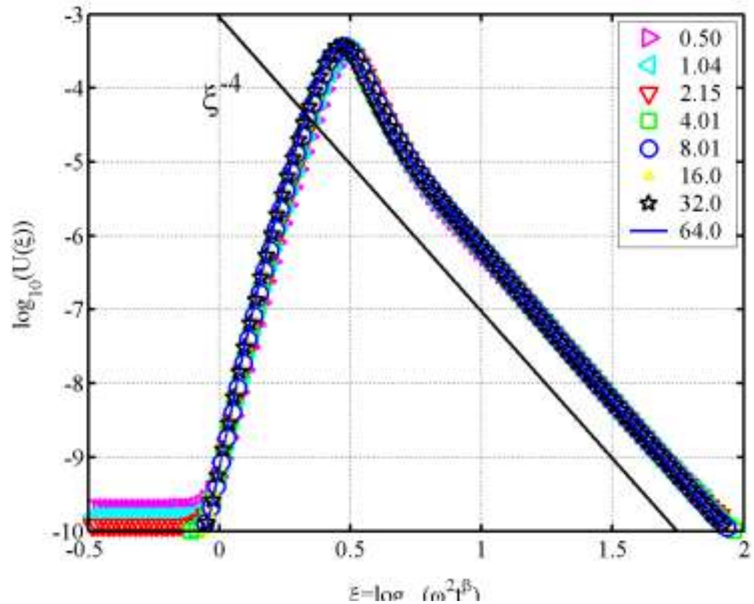
Easy to get in simulations

Sea swell - no input and dissipation

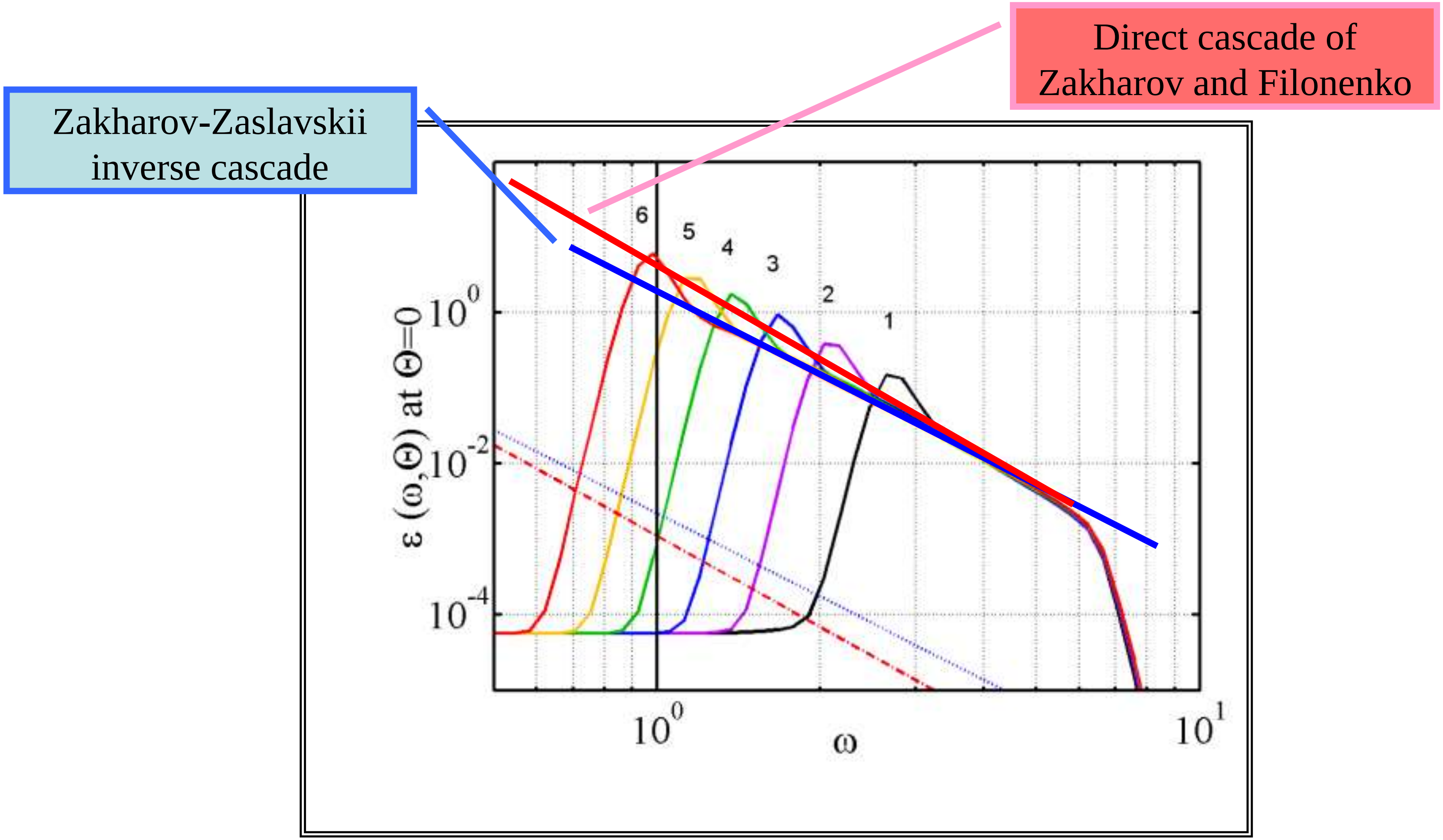
$$n = t^{2/11} U_0(\omega^2 t^{1/11}, \Theta)$$







Growing wind sea
Self-similarity in an explicit form



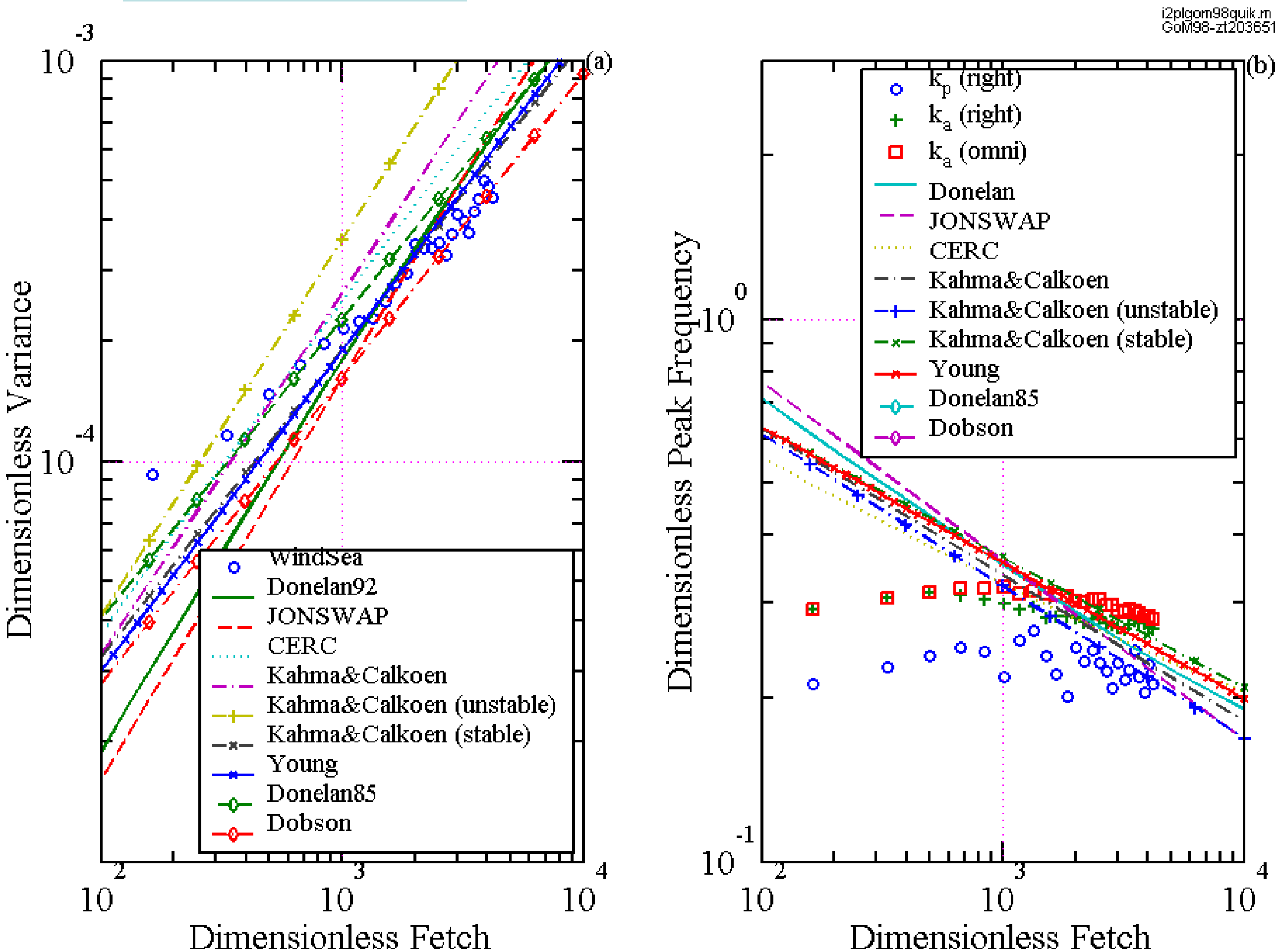
`Scientific curves' of wave growth: wind speed scaling

$$\tilde{\varepsilon} = \tilde{\varepsilon}_0 \chi^{p}$$

$$\tilde{\omega} = \tilde{\omega}_0 \chi^{-q}$$

$0.6 < p < 1.1$; $0.68 < 10^7 \varepsilon_0 < 18.6$;

$0.23 < q < 0.33$; $10.4 < \omega_0 < 22.6$



$$\tilde{\varepsilon}, \tilde{\omega}, p, q$$

are not universal

$$\chi = xg / U_{10}^{2};$$

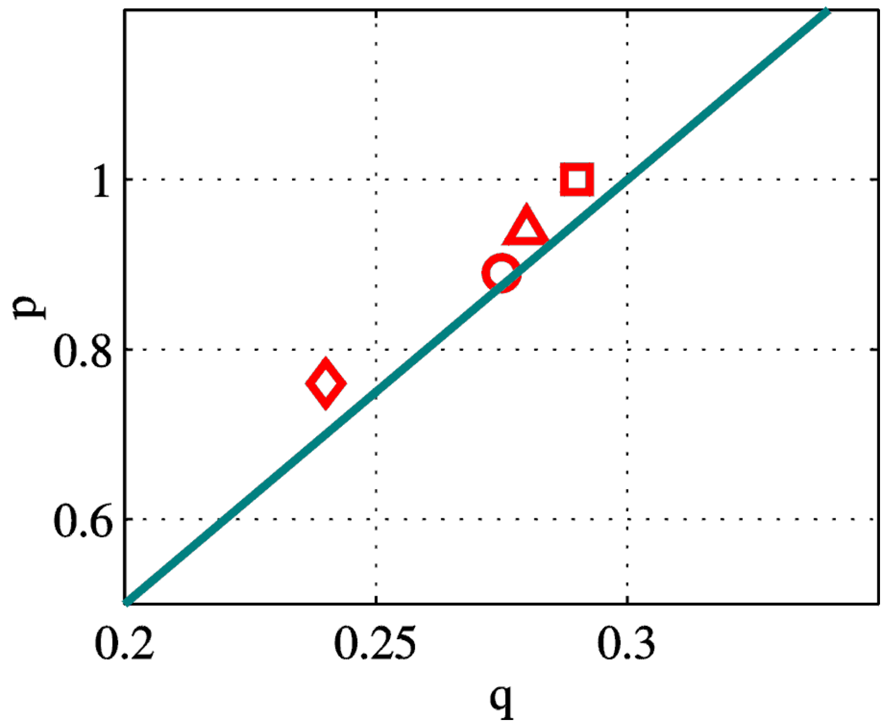
$$\tilde{\varepsilon} = \varepsilon g^{2} / U_{10}^{4};$$

$$\tilde{\omega} = \omega_p U_{10} / g$$

Our thanks to Paul Hwang

`Magic links' in sea experiments

$$\boxed{p = \frac{10q - 1}{2}}$$



○ Black Sea

Babanin et al., 1996

□ US coast, N.Atlantic

Walsh et al 1989

△ Bothnian Sea, unstable

Kahma & Calkoen 1992

◇ Bothnian Sea, stable

Kahma & Calkoen 1992

The `most analytical' theory

`Magic links' of our power-law self-similar solutions can be re-written in a form of simple algebraic relationship

$$\boxed{\mu^4 \nu = \alpha_0^3}$$

$\mu = \langle a k_p \rangle$ - wave steepness

$\nu = \omega_p t \quad (2k_p x)$ - number of waves in periods or wave lengths

$\alpha_0 \approx 0.7$ – a universal constant

Invariant of wave growth

$$\boxed{\mu^4 \nu = \alpha_0^3}$$

- Does not contain wind speed (?!);
- Does not contain parameters of self-similar solutions (parameter of adiabaticity if we assume the slowly varying wave growth conditions);
- Does not refer to initial state. Waves forget their history

How to treat the invariant?

$$\boxed{\mu^4 \nu = \alpha_0^3}$$

- Lifetime is proportional to the instant nonlinear relaxation rate $\nu \sim \mu^{-4} \sim \tau_{nl}$
- In fact, we change a concept:

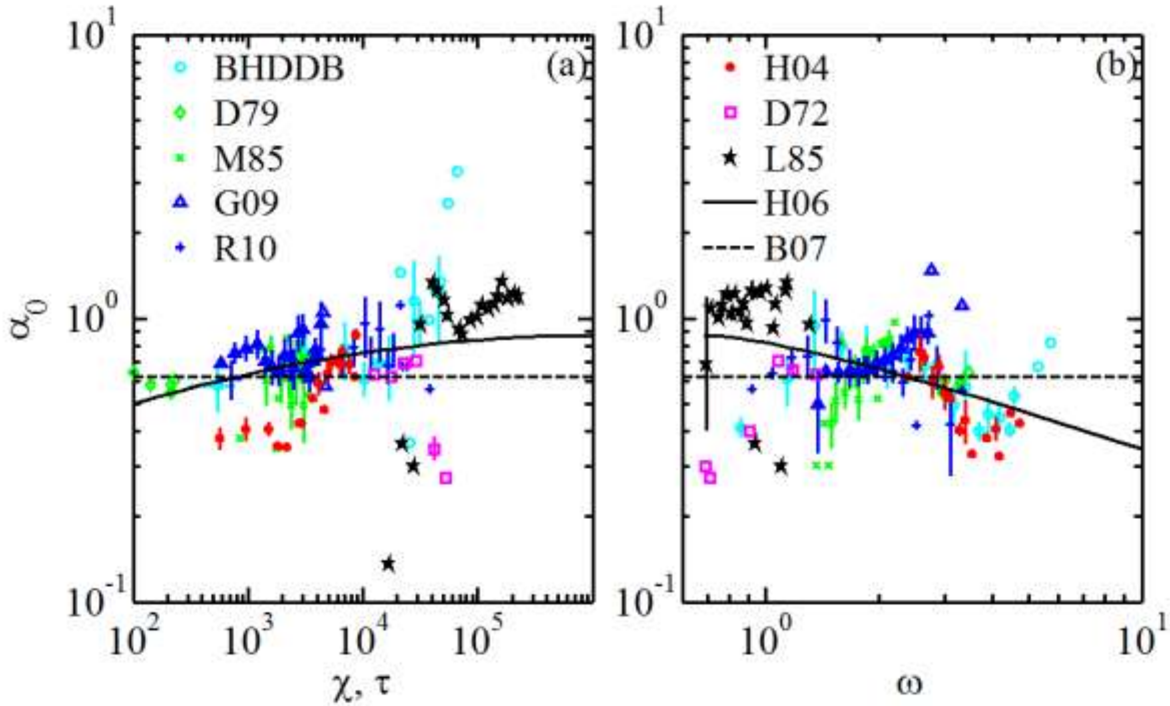
`Waves evolve on their own'

instead of

`Wind rules waves'

Does the invariant work?

Collection of Paul A. Hwang of sea experiments

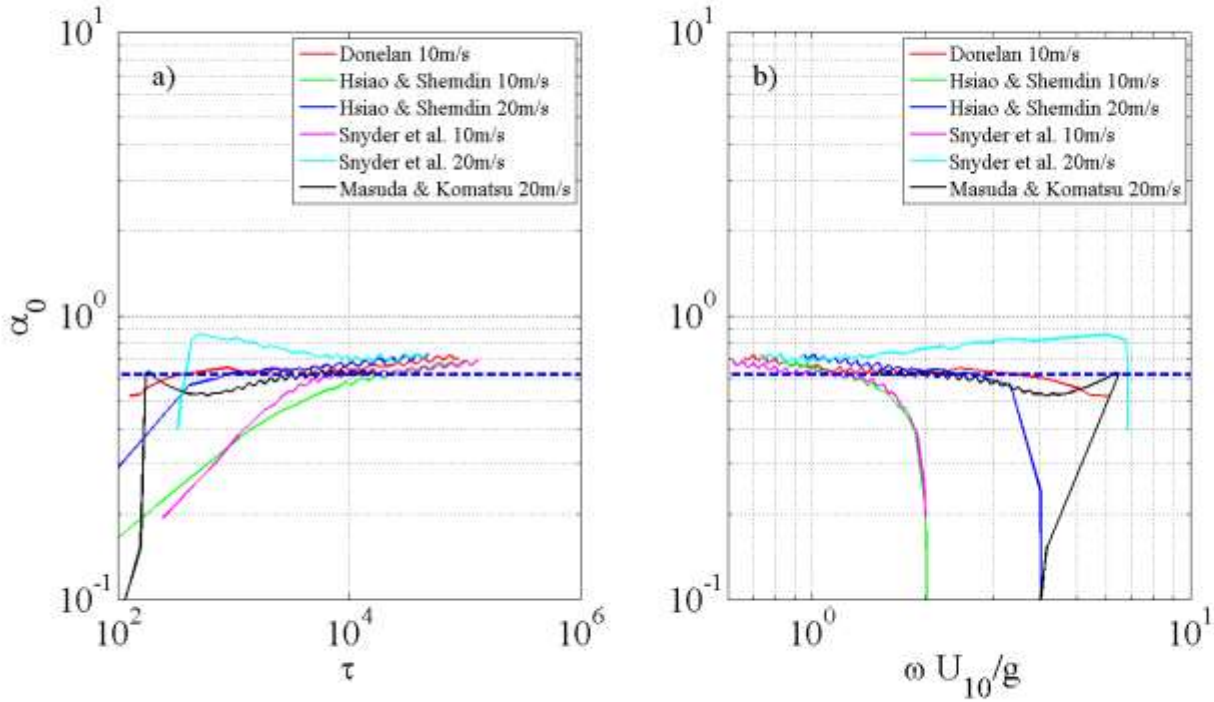


and his `empirical invariant' $(\mu^4 \nu)\chi^{-0.54+0.039\ln\chi} = \alpha_{empirical}$

$\alpha_{empirical}$ varies *e*-times for 5 orders of dimensionless fetch !!!

Does the invariant work?

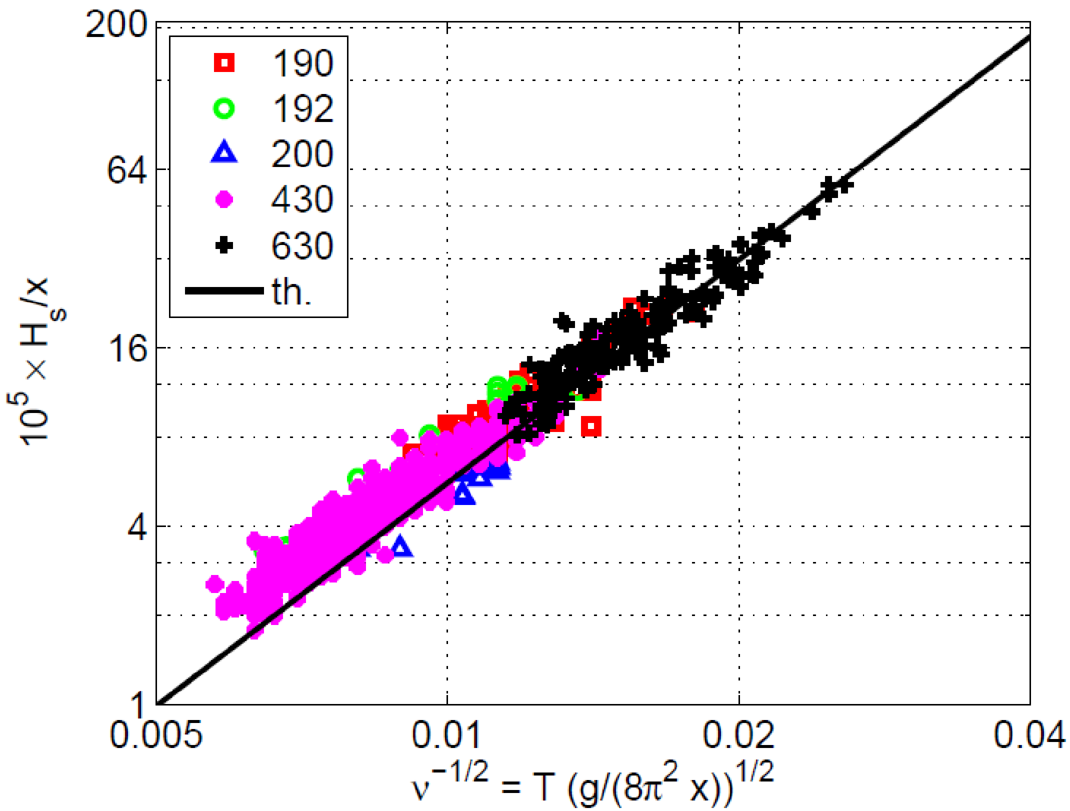
Our collection of simulations of duration-limited growth



Somewhat eclectic presentation: wind-free invariant (ordinate) vs wind speed scaled variables (absise)

Our wind-free invariant implies wind-free scaling of wave growth dependencies

$$\tilde{H} = {H_s}\big/{Fetch}\;; \quad \tilde{T} = T\sqrt{g\big/(8\pi^2 Fetch)}$$



$$\tilde{H} \sim \tilde{T}^{5/2}$$

Waves in a sector $\pm 30^0$ to the off-shore direction for up to 15 years of measurements